Differential Equations Final

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(12 pts) 1. Find the solution of the given initial value problem

\[ y'' + 4y = 2t^2 + 5e^t, \quad y(0) = 1, \quad y'(0) = -2 \]

Sol. The characteristic equation for the homogeneous problem is

\[ r^2 + 4 = 0, \] with complex roots \( r = \pm 2i \). Hence the general solution for the homogeneous problem is

\[ y_c(t) = c_1 \cos 2t + c_2 \sin 2t + \]

Let \( L[y] := y'' - 2y' + y \). Assume that the particular solution \( Y(t) \) of the problem \( L[y] = 2t^2 + 5e^t \) is of the form \( Y(t) = (At^2 + Bt + C) + De^t \). Substituting \( Y \) into the equation, we get

\[ 4At^2 + 4Bt + 2(A + 2C) + 5De^t = 2t^2 + 5e^t \]

Then we obtain the system of equations

\[ 4A = 2, \quad 4B = 0, \quad A + 2C = 0, \quad 5D = 1 \]

which implies \( A = \frac{1}{2}, \quad C = -\frac{1}{4} \) and \( D = 1 \). Thus \( Y(t) = e^t + \frac{t^2}{2} - \frac{1}{4} \). Therefore, the general solution for the equation is

\[ y(t) = y_c(t) + Y(t) \]

\[ = c_1 \cos 2t + c_2 \sin 2t + e^t + \frac{t^2}{2} - \frac{1}{4} \]

Then \( y'(t) = 2c_2 \cos 2t - 2c_1 \sin 2t + e^t + t \). Since \( y(0) = 1, \quad y'(0) = -2 \), we have that

\[ c_1 + \frac{3}{4} = 1, \quad 2c_2 + 1 = -2 \]
which shows that \( c_1 = -3, \ c_2 = 4. \) Hence the solution for the initial value problem is

\[
y(t) = \frac{1}{4} \cos 2t - \frac{3}{2} \sin 2t + e^t + \frac{t^2}{2} - \frac{1}{4}
\]

(18 pts) 2. (1) Find the general solution of the equation

\[
y'' + 6y' + 9y = t^{-1}e^{-3t}, \quad t > 0
\]

(2) Use the method of reduction of order to solve the equation

\[
ty'' - (1 + t)y' + y = t^3 e^{-t}, \quad t > 0 \text{ with } y_1(t) = 1 + t
\]

Sol.

(a) The characteristic equation for the homogeneous problem is \( r^2 + 6r + 9 = 0, \) with real root \( r = -3 \) of multiplicity 2. Hence the general solution for the homogeneous problem is

\[
y_c(t) = c_1 e^{-3t} + c_2 te^{-3t}
\]

Let \( y_1(t) = e^{-3t}, \ y_2(t) = te^{-3t}, \) then \( W(y_1, y_2)(t) = e^{-6t}. \) The particular solution is given by \( Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \) in which

\[
u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)} dt = -\int dt = -t
\]

\[
u_2(t) = \int \frac{y_1(t)g(t)}{W(t)} dt = \int \frac{dt}{t} = \ln t
\]

since \( t > 0. \) Hence \( Y(t) \) is of the form

\[
Y(t) = t(\ln t - 1)e^{-3t}
\]

and the general solution for the equation is

\[
y(t) = c_1 e^{-3t} + c_2 te^{-3t} + (t \ln t)e^{-3t}
\]
(b) Write the equation as the form

\[ y'' - \frac{1 + t}{t} y' + \frac{1}{t} y = t^2 e^{-t} \]

It is easy to see that \( y_1'' - t^{-1}(1 + t)y_1' + t^{-1}y_1 = 0 \). Set \( y = y_1v \), then \( v \) satisfies the equation

\[ (1 + t)v'' + \left( 2 - \frac{(1 + t)^2}{t} \right)v' = t^2 e^{-t} \]

that is, \( v'' - t^{-1}(1 + t)^{-1}(t^2 + 1)v' = t^2(1 + t)^{-1}e^{-t} \). By using the integrating factor

\[ \mu(t) = \exp \left( - \int \frac{t^2 + 1}{t(1 + t)} dt \right) = \exp \left( \int \left( \frac{2}{1 + t} - \frac{1}{t} - 1 \right) dt \right) = \frac{(1 + t)^2 e^{-t}}{t} \]

since \( t > 0 \). We can solve \( v' \) as

\[ v'(t) = \frac{te^t}{(1 + t)^2} \int^t \frac{(1 + s)^2 e^{-s}}{s} \cdot \frac{s^2 e^{-s}}{1 + s} ds = \frac{te^t}{(1 + t)^2} \int^t s(1 + s)e^{-2s} ds \]

\[ = \frac{te^t}{(1 + t)^2} \left( - \frac{(1 + t)^2 e^{-2t}}{2} + c_1 \right) = -\frac{te^{-t}}{2} + c_1 \frac{te^t}{(1 + t)^2} \]

for some constant \( c_1 \). Hence

\[ v(t) = -\frac{1}{2} \int^t se^{-s} ds + c_1 \int^t \frac{se^s}{(1 + s)^2} ds \]

\[ = \frac{1 + t}{2} e^{-t} + c_1 \frac{e^t}{1 + t} + c_2 \]

for some constant \( c_2 \). Therefore, the general solution for the equation is

\[ y(t) = y_1(t)v(t) = c_1 e^t + c_2 (1 + t) + \frac{(1 + t)^2}{2} e^{-t} \]

□
(15 pts) 3. Use the method of reduction of order to solve the equation

\[(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, \quad t < 2; \quad y_1(t) = e^t\]

**Sol.** Write the equation in the form

\[y''' + \frac{2t-3}{2-t}y'' - \frac{t}{2-t}y' + \frac{1}{2-t}y = 0\]

Assume that \(y(t)\) is a solution of the equation with \(y(t) = y_1(t)v(t) = e^tv(t)\), then \(v(t)\) satisfies the equation

\[e^tv''' + \left(3e^t + e^t\frac{2t-3}{2-t}\right)v'' + \left(3e^t + e^t\frac{2t-3}{2-t} - e^t\frac{t}{2-t}\right)v' = 0\]

That is, by letting \(w = v''\), we have that

\[w' + \frac{t-3}{t-2}w = 0\]

which can be solved that

\[v''(t) = w(t) = c_1(t-2)e^{-t}\]

Hence

\[v(t) = c_1te^{-t} + c_2 + c_3\]

Hence the general solution for the equation is of the form

\[y(t) = e^tv(t) = c_1t + c_2te^t + c_3e^t\]

\(\square\)

(14 pts) 4. Find the general solution of the given equations.

(1) \(y^{(5)} - 8y'' = 0.\) \hspace{1cm} (2) \(y^{(8)} + 8y^{(4)} + 16y = 0\)

**Sol.**

(1) The characteristic equation is \(r^5 - 8r^2 = r^2(r^3 - 2)(r^5 + 2r + 4) = 0\), which has roots \(r = 2, \ -1 \pm \sqrt{3}i\) and \(r = 0\) with multiplicity 2. Therefore the general solution is of the form

\[y(t) = c_1 + c_2t + c_3e^{2t} + e^{-t}(c_4 \cos \sqrt{3}t + \sin \sqrt{3}t)\]
(2) The characteristic equation is 
\[ r^8 + 8r^4 + 16r = (r^4 + 4)^2 = 0, \]
which has roots \( r = 1 \pm i, \ -1 \pm i \) with multiplicity 2 for each respectively. Therefore the general solution is of the form 
\[
y(t) = e^t \left( c_1 \cos t + c_2 \sin t \right) + te^t \left( c_3 \cos t + c_4 \sin t \right) \\
+ e^{-t} \left( c_5 \cos t + c_6 \sin t \right) + te^{-t} \left( c_7 \cos t + c_8 \sin t \right)
\]
\( \square \)

(15 pts) 5. Determine a suitable form for \( Y(t) \) if the method of undetermined coefficients is to be used. Do not evaluate the constants.

(1) \( y''' - 2y'' + y = t^3 + 2e^t \).

(2) \( y''' - y' = te^{-t} + 2 \sin t \).

(3) \( y^{(4)} - y''' - y'' + y' = t^2 + 4 + t \sin t \).

Sol.

(1) The characteristic equation for the homogeneous problem is \( r^3 - 2r^2 + r = r(r - 1)^2 = 0 \), with roots \( r = 0 \) and \( r = 1 \) of multiplicity 2. Hence the general solution for the homogeneous problem is 
\[
y_c(t) = c_1 + c_2 e^t + c_3 t e^t
\]

Let \( L[y] := y''' - 2y'' + y' \). If \( Y(t) \) is the particular solution of the problem \( L[y] = t^3 + 2e^t \), then it can be assumed as the form 
\[
Y(t) = t(A_3 t^3 + A_2 t^2 + A_1 t + A_0) + B t^2 e^t
\]

(2) The characteristic equation for the homogeneous problem is \( r^3 - r = r(r - 1)(r + 1) = 0 \), with roots \( r = 0, \pm 1 \). Hence the general solution for the homogeneous problem is 
\[
y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}
\]

Let \( L[y] := y''' - y' \). If \( Y(t) \) is the particular solution of the problem \( L[y] = te^{-t} + 2 \sin t \), then it can be assumed as the form 
\[
Y(t) = t(At + B)e^{-t} + C \cos t + D \sin t
\]

(3) The characteristic equation for the homogeneous problem is \( r^4 - r^3 - r^2 + r = r(r + 1)(r - 1)^2 = 0 \), with roots \( r = 0, -1 \) and
$r = 1$ with multiplicity 2. Hence the general solution for the homogeneous problem is

$$y_c(t) = c_1 + c_2 e^{-t} + (c_3 + c_4 t) e^t$$

Let $L[y] := y^{(4)} - y''' - y'' + y'$. If $Y(t)$ is the particular solution of the problem $L[y] = t^2 + 4 + t \sin t$, then it can be assumed as the form

$$Y(t) = t(At^2 + Bt + C) + (Dt + E) \sin t + (Ft + G) \cos t$$

(16 pts) 6. Find the general solution for the following equations

(1) Verify that $x$, $x^2$, and $x^3$ are solutions of the homogeneous equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$$

(2) Find a formula involving integrals for the particular solution of the equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = g(x), \quad x > 0.$$  

Sol.

(1) Let $y_1(x) = x$, $y_2(x) = x^2$, $y_3(x) = x^3$. Then

$$x^3 y_1''' - 3x^2 y_1'' + 6xy_1' - 6y_1 = 6x - 6x = 0$$

$$x^3 y_2''' - 3x^2 y_2'' + 6xy_2' - 6y_2 = -6x^2 + 12x^2 - 6x^2 = 0$$

$$x^3 y_3''' - 3x^2 y_3'' + 6xy_3' - 6y_3 = 6x^3 - 18x^3 + 18x^3 - 6x^3 = 0$$

which shows that $y_1$, $y_2$, $y_3$ are indeed solution of the homogeneous equation.
(2) Hence $W(y_1, y_2, y_3)(x) = 2x^3$. Also,

$$
\begin{align*}
W_1(x) &= \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4 \\
W_2(x) &= \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3 \\
W_3(x) &= \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2
\end{align*}
$$

The particular solution is given by $Y(x) = \sum_{i=1}^{3} u_i(x)y_i(x)$, in which

$$
\begin{align*}
    u_1(x) &= \int \frac{g(x)W_1(x)}{W(x)}dx = \frac{1}{2} \int \frac{x(t)}{t^2} dt \\
    u_2(x) &= \int \frac{g(x)W_2(x)}{W(x)}dx = -\int \frac{t}{t^3} dt \\
    u_3(x) &= \int \frac{g(x)W_3(x)}{W(x)}dx = \frac{1}{2} \int \frac{x(t)}{t^4} dt
\end{align*}
$$

Hence $Y(t)$ is of the form

$$
Y(t) = \frac{x}{2} \int \frac{x(t)}{t^2} dt - x^2 \int \frac{t}{t^3} dt + \frac{x^3}{2} \int \frac{x(t)}{t^4} dt
$$

$$
= \int \left( \frac{x}{2t^2} - \frac{x^2}{t^3} + \frac{x^3}{2t^4} \right) g(t) dt
$$

$$
= \int \frac{x(x-t)^2}{2t^4} g(t) dt
$$

(15 pts) 7. The Chebyshev differential equation is

$$
(1 - x^2)y'' - xy' + \alpha^2 y = 0,
$$
where $\alpha$ is a constant. Determine two solutions in powers of $x$ and their radius of convergence.

*Hint:* $x = 0$ is a regular point of the equation.

*Sol.* Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substituting into the equation we have

$$(1-x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

It follows that

$$(2a_2 + \alpha^2 a_0) + (6a_3 + (\alpha^2 - 1)a_1)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + (\alpha^2 - n^2)a_n] x^n = 0$$

We obtain the recurrence relation

$$a_{n+2} = \frac{n^2 - \alpha^2}{(n+2)(n+1)} a_n, \quad n = 0, 1, 2, \ldots$$

According to the recurrence relation we have that

$$a_2 = -\frac{\alpha^2}{2!} a_0, \quad a_4 = \frac{2^2 - \alpha^2}{4 \cdot 3} a_2 = -\frac{\alpha^2(2^2 - \alpha^2)}{4!} a_0, \ldots$$

$$a_3 = \frac{1^2 - \alpha^2}{3!} a_1, \quad a_5 = \frac{3^2 - \alpha^2}{5 \cdot 4} a_5 = \frac{(3^2 - \alpha^2)(1^2 - \alpha^2)}{5!} a_1, \ldots$$

That is,

$$a_{2m} = -\frac{\alpha^2(2^2 - \alpha^2) \cdots ((2m-2)^2 - \alpha^2)}{(2m)!} a_0, \quad m = 1, 2, \ldots$$

$$a_{2m+1} = \frac{(1^2 - \alpha^2)(3^2 - \alpha^2) \cdots ((2m-1)^2 - \alpha^2)}{(2m+1)!} a_1, \quad m = 1, 2, \ldots$$
Therefore we get two linearly independent solutions by setting \((a_0, a_1) = (1, 0)\) and \((a_0, a_1) = (0, 1)\) respectively:

\[
y_1(x) = 1 - \sum_{m=1}^{\infty} \frac{\alpha^2(2^2 - \alpha^2) \cdots ((2m - 2)^2 - \alpha^2)}{(2m)!} x^{2m}\]

\[
y_2(x) = x + \sum_{m=1}^{\infty} \frac{(1^2 - \alpha^2)(3^2 - \alpha^2) \cdots ((2m - 1)^2 - \alpha^2)}{(2m + 1)!} x^{2m+1}\]

For \(y_1(x)\), let \(a_m = \frac{\alpha^2(2^2 - \alpha^2) \cdots ((2m-2)^2 - \alpha^2) x^{2m}}{(2m)!}\), then

\[
\lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \left| \frac{\alpha^2(2^2 - \alpha^2) \cdots ((2m)^2 - \alpha^2) x^{2m+2}}{(2m+2)!} \cdot \frac{(2m)!}{\alpha^2(2^2 - \alpha^2) \cdots ((2m-2)^2 - \alpha^2) x^{2m}} \right|
\]

\[
= \lim_{m \to \infty} \left| \frac{(2m)^2 - \alpha^2}{(2m + 2)(2m + 1)} x^2 \right|
\]

\[
= |x|^2 \left| \lim_{m \to \infty} \frac{(2m)^2 - \alpha^2}{(2m + 2)(2m + 1)} \right| = |x|^2
\]

This shows that the series solution \(y_1(x)\) converges if \(|x|^2 < 1\) and thus the radius of convergence for \(y_1(x)\) is 1. For \(y_2(x)\), let \(b_m = \frac{(1^2 - \alpha^2)(3^2 - \alpha^2) \cdots ((2m-1)^2 - \alpha^2) x^{2m+1}}{(2m+1)!}\), then

\[
\lim_{m \to \infty} \left| \frac{b_{m+1}}{b_m} \right| = \lim_{m \to \infty} \left| \frac{(1^2 - \alpha^2)(3^2 - \alpha^2) \cdots ((2m+1)^2 - \alpha^2) x^{2m+3}}{(2m+3)!} \cdot \frac{(2m)!}{(1^2 - \alpha^2)(3^2 - \alpha^2) \cdots ((2m-1)^2 - \alpha^2) x^{2m+1}} \right|
\]

\[
= \lim_{m \to \infty} \left| \frac{(2m + 1)^2 - \alpha^2}{(2m + 3)(2m + 2)} x^2 \right|
\]

\[
= |x|^2 \left| \lim_{m \to \infty} \frac{(2m + 1)^2 - \alpha^2}{(2m + 3)(2m + 2)} \right| = |x|^2
\]

This shows that the series solution \(y_2(x)\) converges if \(|x|^2 < 1\) and thus the radius of convergence for \(y_2(x)\) is also 1.

\[
8. \text{The Bessel equation of order one is} \quad x^2 y'' + xy' + (x^2 - 1)y = 0.
\]
(a) Show that $x = 0$ is a regular singular point.
(b) Show that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$.
(c) Construct the series solution at $x = 0$ for $r = 1$.
(d) Show that the series solution converges for all $x$.

*Sol.*

(a) Rewrite the equation of the form

$$y'' + \frac{1}{x} y' + \frac{x^2 - 1}{x^2} y = 0$$

and let $p(x) = \frac{1}{x}$, $q(x) = \frac{x^2 - 1}{x^2}$. The only singular points is $x = 0$.

At $x = 0$,

$$p_0 = \lim_{x \to 0} x p(x) = \lim_{x \to 0} 1 = 1$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 1} (x^2 - 1) = -1$$

Hence $x = 0$ is a regular singular point.

(b) Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

Substituting into the equation we have

$$\sum_{n=0}^{\infty} [(n + r)^2 - 1] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

It follows that

$$(r^2 - 1)a_0 x^r + [(r+1)^2 - 1]a_1 x^{r+1} + \sum_{n=2}^{\infty} [(n+r)^2 - 1] a_n + a_{n-2}) x^{n+r} = 0$$

The indicial equation is $r^2 - 1 = 0$ with distinct real roots $r_1 = 1$, $r_2 = -1$. 

(c) The series solution at $x = 0$ for $r = 1$.

(d) Show that the series solution converges for all $x$. 

(c) For either \( r = \pm 1 \), it is necessary to take \( a_1 = 0 \) in order that the coefficient of \( x^{r+1} \) be zero. Note that the recurrence relation is of the form
\[
a_n = -\frac{a_{n-2}}{(n+r)^2 - 1}, \quad n = 2, 3, \ldots
\]
The fact that \( a_1 = 0 \) implies that \( a_{2m+1} = 0, \quad \forall \ m = 0, 1, 2, \ldots \). If \( r = 1 \), then
\[
a_{2m} = -\frac{a_{2m-2}}{(2m+1)^2 - 1} = -\frac{a_{2m-2}}{4m(m+1)} = \frac{a_{2m-4}}{4^2m^2(m+1)(m-1)}
\]
\[
= -\frac{a_{2m-6}}{4^3m(m+1) \cdot (m-1)m \cdot (m-2)(m-1)}
\]
\[
= \cdots = \frac{(-1)^m a_0}{4^m m!(m+1)!}
\]
By setting \( a_0 = \frac{1}{2} \), we obtain one solution for \( x > 0 \) of the equation
\[
J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!(n+1)!}.
\]

(d) Let \( a_n = \frac{(-1)^n}{2^n n!(n+1)!}, \quad n = 0, 1, \ldots \). Then
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{2^{2n+2} (n+1)! (n+2)!} \right| = \lim_{n \to \infty} \frac{1}{4(n+1)(n+2)} = 0
\]
which shows that the radius of convergence of \( J_0(x) \) is infinite, that is, \( J_1(x) \) converges for all \( x \).

\( \square \)