ODE Homework 2

2.6. Exact Equations and Integrating Factors

1. Determine whether the following equation is exact. If it is exact, find the solution

\[(e^x \sin y + 2y)dx - (3x - e^x \sin y)dy = 0\]

[§2.6 #8]
Sol. Let \(M(x, y) = e^x \sin y + 2y,\ N(x, y) = e^x \sin y - 3x\), then

\[M_y = e^x \cos y + 2, \quad N_x = e^x \sin y - 3\]

Since \(M_y \neq N_x\), the equation is not exact. \(\square\)

2. Determine whether the following equation is exact. If it is exact, find the solution

\[\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0, \quad x > 0\]

[§2.6 #11]
Sol. Let \(M(x, y) = \frac{y}{x} + 6x,\ N(x, y) = \ln x - 2\), then

\[M_y = N_x = \frac{1}{x}\]

which shows that the equation is exact. Thus there exists a function \(\psi(x, y)\) such that \(\psi_x = M, \ \psi_y = N\). Integrating \(M(x, y)\) with respect to \(x\), we get

\[\psi(x, y) = \int \left(\frac{y}{x} + 6x\right)dx + g(y) = 3x^2 + y \ln x + g(y)\]

for some function \(g(y)\). Now \(\psi_y = \ln x + g'(y) = \ln x - 2\), so \(g(y) = -2y - c_0\) for some constant \(c_0\). Hence the solution is given implicitly as \(\psi(x, y) = C\), i.e.,

\[3x^2 + y \ln x - 2y = C\]

\(\square\)

3. Determine whether the following equation is exact. If it is exact, find the solution

\[\frac{x\ dx}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{y\ dy}{(x^2 + y^2)^{\frac{3}{2}}} = 0\]
§2.6 #12
Sol. Let \( M(x, y) = \frac{x}{(x^2 + y^2)^{\frac{3}{2}}} \), \( N(x, y) = \frac{y}{(x^2 + y^2)^{\frac{3}{2}}} \), then

\[
M_y = N_x = \frac{-3xy}{(x^2 + y^2)^{\frac{3}{2}}}
\]

which shows that the equation is exact. Thus there exists a function \( \psi(x, y) \) such that \( \psi_x = M, \psi_y = N \). Integrating \( M(x, y) \) with respect to \( x \), we get

\[
\psi(x, y) = \int \frac{x \, dx}{(x^2 + y^2)^{\frac{3}{2}}} + g(y) = \frac{-1}{\sqrt{x^2 + y^2}} + g(y)
\]

for some function \( g(y) \). Now \( \psi_y = \frac{y}{(x^2 + y^2)^{\frac{3}{2}}} + g'(y) = \frac{y}{(x^2 + y^2)^{\frac{3}{2}}} \), so \( g(y) = c_0 \) for some constant \( c_0 \). Hence the solution is given implicitly as \( \psi(x, y) = C \), i.e.,

\[
x^2 + y^2 = C
\]

4. Solve the initial value problem

\[
(9x^2 + y - 1) \, dx - (4y - x) \, dy = 0, \quad y(1) = 0
\]

and determine at least approximately where the solution is valid.

§2.6 #14
Sol. Let \( M(x, y) = 9x^2 + y - 1, \) \( N(x, y) = x - 4y \), then

\[
M_y = N_x = 1
\]

which shows that the equation is exact. Thus there is a function \( \psi(x, y) \) such that \( \psi_x = M, \psi_y = N \). Integrating \( M(x, y) \) with respect to \( x \), we get

\[
\psi(x, y) = \int (9x^2 + y - 1) \, dx + g(y) = 3x^3 + xy - x + g(y)
\]

for some function \( g(y) \). Now \( \psi_y = x + g'(y) = x - 4y \), so \( g(y) = -2y^2 + c_0 \) for some constant \( c_0 \). Hence the solution is given implicitly as \( \psi(x, y) = C \), i.e.

\[
3x^3 + xy - x - 2y^2 = C
\]

Since \( y(1) = 0 \), we have \( C = 3 - 1 = 2 \), that is, the solution for the initial value problem is defined implicitly by the equation

\[
3x^3 + xy - x - 2y^2 = 2
\]
In fact, according to the initial condition, \( y \) can be expressed explicitly as a function of \( x \), which is

\[
2y^2 - xy - (3x^3 - x - 2) = 0
\]

\[
y(x) = \frac{x - \sqrt{x^2 + 8(3x^3 - x - 2)}}{4} = \frac{x - \sqrt{24x^3 + x^2 - 8x - 16}}{4}
\]

It is clear that the solution is defined for the region of \( x \) where \( 24x^3 + x^2 - 8x - 16 > 0 \). Let \( P(x) = 24x^3 + x^2 - 8x - 16 \), note that the discriminant for \( P(x) \) is \( \Delta = -3876736 < 0 \) which implies that \( P(x) = 0 \) has only one real root. By Solving \( P(x) = 0 \), we have that \( x \approx 0.984587 \). Since \( P(0) = -16 < 0 \), we can conclude that the solution for the initial value problem is valid for \( x > 0.984587 \). \( \square \)

5. Show that the equation

\[
\left( \frac{\sin y}{y} - 2e^{-x} \sin x \right) dx + \left( \frac{\cos y + 2e^{-x} \cos x}{y} \right) dy = 0
\]

is not exact but becomes exact when multiplied the integrating factor \( \mu(x, y) = ye^x \).

[S2.6 #21]

Sol. Let \( M(x, y) = \frac{\sin y}{y} - 2e^{-x} \sin x, \ N(x, y) = \frac{\cos y + 2e^{-x} \cos x}{y} \).

Then

\[
M_y = \frac{y \cos y - \sin y}{y^2}, \quad N_x = -\frac{2e^{-x} \cos x - 2e^{-x} \sin x}{y}
\]

which shows that the equation is not exact. Now multiple the equation by \( \mu(x, y) = ye^x \), we have that

\[
(e^x \sin y - 2y \sin x) dx + (e^x \cos y + 2 \cos x) dy = 0
\]

Note that

\[
(\mu M)_y = (\mu N)_x = e^x \cos y - 2 \sin x
\]

which shows that the equation \( \mu M dx + \mu N dy = 0 \) is exact. Thus there exists a function \( \psi(x, y) \) such that \( \psi_x = \mu M, \ \psi_y = \mu N \).

Integrating \( \mu(x, y)M(x, y) \) with respect to \( x \), we get

\[
\psi(x, y) = \int (e^x \sin y - 2y \sin x) dx + g(y) = e^x \sin y + 2y \cos x + g(y)
\]

for some function \( g(y) \). Now \( \psi_y = e^x \cos y + 2 \cos x + g'(y) = e^x \cos y + 2 \cos x, \) so \( g(y) = c_0 \) for some constant \( c_0 \). Hence the solution is given implicitly as \( \psi(x, y) = C \), i.e.,

\[
e^x \sin y + 2y \cos x = C
\]
6. Show that if \((N_x - M_y)/(xM - yN) = R\), where \(R\) depends on
the quantity \(xy\) only, then the differential equation
\[ M + Ny' = 0 \]
has an integrating factor of the form \(\mu(xy)\). Find a general
formula for this integrating factor.

\[\text{Proof.}\] Multiply the original equation by a function \(\mu = \mu(x, y)\).
The equation \(\mu M + \mu Ny' = 0\) has an integrating factor if
\((\mu M)_y = (\mu N)_x\), that is,
\[ \mu_y M - \mu_x N = \mu N_x - \mu M_y = \mu(N_x - M_y) \]
Define \(R := \frac{N_x - M_y}{xM - yN}\), then \(N_x - M_y = R(xM - yN)\). If \(R\) is some
function depending only on the quantity \(z := xy\). It follows that
the modified form of the equation is exact, if
\[ \mu_y M - \mu_x N = \mu R(xM - yN) = x\mu RM - y\mu RN \]
This relation is satisfied if
\[ \mu_y = x\mu R \text{ and } \mu_x = y\mu R \]
Now consider \(\mu = \mu(z) = \mu(xy)\). Then the partial derivatives
are \(\mu_x = y\mu'\) and \(\mu_y = x\mu'\) with \(\mu' = \frac{du}{dz}\). Thus
\[ x\mu' = x\mu R \text{ and } y\mu' = y\mu R \]
which implies that \(\mu\) must satisfy \(\mu'(z) = \mu(z)R(z)\). The later
equation is separable, with \(\frac{du}{u} = Rdz\). Hence
\[ \mu(z) = \exp \left( \int R(z)dz \right) \]
Therefore, given \(R = R(xy)\), it is possible to determine \(\mu = \mu(xy)\) which becomes an integrating factor of the
differential equation. \(\square\)

7. Find an integrating factor and solve the equation
\[ e^x dx + (e^x \cot y + 2y \csc y) dy = 0 \]

\[\text{Sol.}\] Let \(M(x, y) = e^x, N(x, y) = e^x \cot y + 2y \csc y\). Then
\[ M_y = 0, \quad N_x = e^x \cot y \]
which shows that the equation is not exact. However, since
\[
\frac{N_x - M_y}{M} = \frac{e^x \cot y - 0}{e^x} = \cot y
\]
which is only depend on \(y\), so the equation has an integrating factor \(\mu = \mu(y)\). Furthermore, we have that
\[
\mu(y) = \exp \left( \int \cot s \, ds \right) = \sin y
\]
Multiple the equation by \(\mu(y) = \sin y\), we get the modified equation
\[
e^x \sin y \, dx + (e^x \cos y + 2y) \, dy = 0
\]
which is exact. Let \(M(x, y) = e^x \sin y\), \(N(x, y) = e^x \cos y + 2y\). Then there exists a function \(\psi(x, y)\) such that \(\psi_x = M\), \(\psi_y = N\). Integrating \(\overline{M}\) with respect to \(x\), we get
\[
\psi(x, y) = \int e^x \sin y \, dx + g(y) = e^x \sin y + g(y)
\]
for some function \(g(y)\). Now \(\psi_y = e^x \cos y + g'(y) = e^x \cos y + 2y\), so \(g(y) = y^2 + c_0\) for some constant \(c_0\). Hence the solution is given implicitly as \(\psi(x, y) = C\), i.e.,
\[
e^x \sin y + y^2 = C
\]

2.7. Numerical Approximations: Euler’s Method

8. Find approximate values of the solution of the given initial value problem
\[
y' = 3 + t - y, \quad y(0) = 1
\]
at \(t = 0.1, 0.2, 0.3, \) and 0.4 using the Euler method with \(h = 0.1\).

[\S 2.7 \#1(a)]

Sol. Let \(f(t, y) = 3 + t - y\). The Euler formula is of the form
\[
y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \ldots
\]
where \(f_n = f(t_n, y_n) = 3 + t_n - y_n\). Since \(y(0) = 1\), we set \((t_0, y_0) = (0, 1)\). For \(h = 0.1\), we have that
\[
y_1 = y_0 + (3 + t_0 - y_0)h = 1 + 0.1(3 + 0 - 1) = 1.2
\]
\[
y_2 = y_1 + (3 + t_1 - y_1)h = 1.2 + 0.1(3 + 0.1 - 1.2) = 1.39
\]
\[
y_3 = y_2 + (3 + t_2 - y_2)h = 1.38 + 0.1(3 + 0.2 - 1.38) = 1.571
\]
\[
y_4 = y_3 + (3 + t_3 - y_3)h = 1.542 + 0.1(3 + 0.3 - 1.542) = 1.7439
\]
9. Find approximate values of the solution of the given initial value problem

\[ y' = 3 \cos t - 2y, \quad y(0) = 0 \]

at \( t = 0.1, 0.2, 0.3, \) and 0.4 using the Euler method with \( h = 0.1. \)

\( \text{[§2.7 #4(a)]} \)

**Sol.** Let \( f(t, y) = 3 \cos t - 2y. \) The Euler formula is of the form

\[ y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \ldots \]

where \( f_n = f(t_n, y_n) = 3 \cos t_n - 2y_n. \) Since \( y(0) = 0, \) we set \( (t_0, y_0) = (0, 0). \) For \( h = 0.1, \) we have that

\[
\begin{align*}
y_1 &= y_0 + (3 \cos t_0 - 2y_0) h = 0 + 0.1 (3 \cos 0 - 2 \times 0) = 0.3 \\
y_2 &= y_1 + (3 \cos t_1 - 2y_2) h = 0.3 + 0.1 (3 \cos 0.1 - 2 \times 0.3) \approx 0.5385 \\
y_3 &= y_2 + (3 \cos t_2 - 2y_3) h = 0.54 + 0.1 (3 \cos 0.2 - 2 \times 0.54) \approx 0.7248 \\
y_4 &= y_3 + (3 \cos t_3 - 2y_4) h = 0.732 + 0.1 (3 \cos 0.3 - 2 \times 0.732) \approx 0.8665
\end{align*}
\]

10. **Convergence of Euler’s Method.** It can be shown that, under suitable conditions on \( f, \) the numerical approximation generated by the Euler method for the initial value problem \( y' = f(t, y), \ y(t_0) = y_0 \) converges to the exact solution as the step size \( h \) decreases. This is illustrated by the following example. Consider the initial value problem

\[ y' = 1 - t + y, \quad y(t_0) = y_0. \]

(a) Show that the exact solution is \( y = \phi(t) = \left(y_0 - t_0\right)e^{t-t_0} + t. \)

(b) Using the Euler formula, show that

\[ y_k = (1 + h)y_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \ldots \]

(c) Noting that \( y_1 = (1 + h)(y_0 - t_0) + t_1, \) show by induction that

\[ y_n = (1 + h)^n(y_0 - t_0) + t_n \quad \text{(i)} \]

for each positive integer \( n. \)

(d) Consider a fixed point \( t > t_0 \) and for a given \( n \) choose \( h = \frac{t-t_0}{n}. \) Then \( t_n = t \) for every \( n. \) Note also that \( h \to 0 \) as \( n \to \infty. \) By substituting for \( h \) in Eq. (i) and letting \( n \to \infty, \) show that \( y_n \to \phi(t) \) as \( n \to \infty. \)

**Hint:** \( \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a. \)
Proof.
(a) Note that the equation is $y' - y = 1 - t$ which is linear with integrating is $\mu(t) = \exp(-\int dt) = e^{-t}$. Hence the solution $\phi(t)$ for the initial value problem is of the form

$$\phi(t) = \int_{t_0}^{t} e^{t-s}(1-s)ds + y_0e^{t-t_0}$$

$$= se^{t-s} \bigg|_{s=t_0}^{s=t} + y_0e^{t-t_0} = (y_0 - t_0)e^{t-t_0} + t$$

(b) Let $f(t, y) = 1 - t + y$, then the Euler formula shows that

$$y_k = y_{k-1} + f_{k-1}h = y_{k-1} + hf(t_{k-1}, y_{k-1})$$
$$= y_{k-1} + h(1 - t_{k-1} + y_{k-1})$$
$$= (1 + h)y_{k-1} + h - ht_{k-1}, \ k = 1, 2, \cdots$$

(c) The initial condition $y(t_0) = y_0$ automatically implies

$$y_1 = (1+h)y_0 + h - ht_0 = (1+h)(y_0 - t_0) + t_0 + h = (1+h)(y_0 - t_0) + t_1$$

Assuming that the relation holds for some integer $k \geq 1$, that is,

$$y_k = (1+h)^k(y_0 - t_0) + t_k$$

Then by Euler formula,

$$y_{k+1} = (1 + h)y_k + h - ht_k$$
$$= (1 + h)((1 + h)^k(y_0 - t_0) + t_k) + h - ht_k$$
$$= (1 + h)^{k+1}(y_0 - t_0) + t_k + h - ht_k$$
$$= (1 + h)^{k+1}(y_0 - t_0) + t_k + h$$
$$= (1 + h)^{k+1}(t_0 - t_0) + t_{k+1}$$

Thus, by induction hypothesis,

$$y_n = (1+h)^n(y_0 - t_0) + t_n, \ \forall \ n \in \mathbb{N}$$

(d)

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \left[(1+h)^n(y_0 - t_0) + t_n\right]$$
$$= \lim_{n \to \infty} \left[(1 + \frac{t - t_0}{n})^n(y_0 - t_0) + t\right]$$
$$= (y_0 - t_0) \lim_{n \to \infty} \left(1 + \frac{t - t_0}{n}\right)^n + t$$
$$= (y_0 - t_0)e^{t-t_0} + t = \phi(t)$$
3.1. Homogeneous Equations with Constant Coefficients

11. Find the general solution of the differential equation

\[ y'' + 3y' + 2y = 0 \]

[§3.1 #4]

**Sol.** The characteristic equation is

\[ r^2 + 3r + 2 = (r + 1)(r + 2) = 0 \]

Thus we get \( r = -1, -2 \). Therefore the general solution is of the form

\[ y(t) = c_1 e^{-t} + c_2 e^{-2t} \]

12. Find the general solution of the differential equation

\[ y'' - 9y' + 9y = 0 \]

[§3.1 #7]

**Sol.** The characteristic equation is

\[ r^2 - 9r + 9 = 0 \]

Thus we get \( r = \frac{9 + 3\sqrt{3}}{2} \). Therefore the general solution is of the form

\[ y(t) = c_1 e^{\frac{(9 - 3\sqrt{3})t}{2}} + c_2 e^{\frac{(9 + 3\sqrt{3})t}{2}} \]

13. Find the solution of the initial value problem.

\[ y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0 \]

Sketch the graph of the solution and describe its behavior as \( t \) increases.

[§3.1 #11]

**Sol.** The characteristic equation is

\[ r^2 + 5r + 3 = 0 \]

Thus we get \( r = \frac{-5 \pm \sqrt{13}}{2} \). Therefore the general solution is of the form

\[ y(t) = c_1 e^{\frac{-5 + \sqrt{13}}{2}t} + c_2 e^{\frac{-5 - \sqrt{13}}{2}t} \]
Thus \( y'(t) = \frac{(-5+\sqrt{13})c_1}{2} e^{\frac{-5+\sqrt{13}}{2} t} + \frac{(-5-\sqrt{13})c_2}{2} e^{\frac{-5-\sqrt{13}}{2} t} \). Since \( y(0) = 1, \ y'(0) = 0 \), so

\[
\begin{align*}
c_1 + c_2 &= 1 \\
\frac{-5 - \sqrt{13}}{2} c_1 + \frac{-5 + \sqrt{13}}{2} c_2 &= 0
\end{align*}
\]

and we have that \( c_1 = \frac{13-5\sqrt{13}}{26}, \ c_2 = \frac{13+5\sqrt{13}}{26} \). Hence the solution of the initial value problem is

\[
y(t) = \frac{13 - 5\sqrt{13}}{26} e^{\frac{-5-\sqrt{13}}{2} t} + \frac{13 + 5\sqrt{13}}{26} e^{\frac{-5+\sqrt{13}}{2} t}
\]

Note that both \( \frac{-5+\sqrt{13}}{2} \) are negative, so \( e^{\frac{-5+\sqrt{13}}{2} t} \to 0 \) as \( t \to \infty \). Hence \( \lim_{t \to \infty} y(t) = 0 \). The graph of solution is as follows

14. Find the solution of the initial value problem.

\[
y'' + 8y' - 9y = 0, \ y(1) = 1, \ y'(1) = 0
\]

Sketch the graph of the solution and describe its behavior as \( t \) increases.

[§3.1 #15]

Sol. The characteristic equation is

\[
r^2 + 8r - 9 = 0
\]

Thus we get \( r = -9, \ 1 \). Therefore the general solution is of the form

\[
y(t) = c_1 e^{-9t} + c_2 e^t
\]
Thus \( y'(t) = c_2 e^t - 9c_1 e^{-9t} \). Since \( y(1) = 1 \), \( y'(1) = 0 \), so
\[
e^{-9}c_1 + ec_2 = 1
\]
\[
-9e^{-9}c_1 + ec_2 = 0
\]
and we have that \( c_1 = \frac{e^9}{10} \), \( c_2 = \frac{9}{10e} \). Hence the solution of the initial value problem is
\[
y(t) = \frac{e^9}{10} e^{-9t} + \frac{9}{10e} e^t = \frac{1}{10} e^{9-9t} + \frac{9}{10} e^{t-1}
\]
It is easy to see that
\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left( \frac{e^9}{10} e^{-9t} + \frac{9}{10e} e^t \right) = \frac{9}{10e} \lim_{t \to \infty} e^t = \infty
\]
The graph of solution is as follows

---

15. Solve the initial value problem \( y'' - y' - 2y = 0 \), \( y(0) = \alpha \), \( y'(0) = 2 \). Then find \( \alpha \) so that the solution approaches zero as \( t \to \infty \).

[§3.1 #21]

Sol. The characteristic equation is
\[
r^2 - r - 2 = 0
\]
Thus we get \( r = -1, \ 2 \). Therefore the general solution is of the form
\[
y(t) = c_1 e^{-t} + c_2 e^{2t}
\]
Thus \( y'(t) = 2c_2 e^{2t} - c_1 e^{-t} \). Since \( y(0) = \alpha \), \( y'(0) = 2 \), so
\[
c_1 + c_2 = \alpha
\]
\[
2c_2 - c_1 = 2
\]
and we have that \( c_1 = \frac{2\alpha - 2}{3}, \ c_2 = \frac{\alpha + 2}{3} \). Hence the solution of the initial value problem is

\[
y(t) = \frac{2\alpha - 2}{3} e^{-t} + \frac{\alpha + 2}{3} e^{2t}
\]

In order to make \( y(t) \to 0 \) as \( t \to \infty \), it suffice to set \( \alpha \) such that the coefficient of \( e^{2t} \) to be zero. Thus, by letting \( \frac{\alpha + 2}{3} = 0 \Rightarrow \alpha = -2 \), we have that \( y(t) = -2e^{-t} \), and \( \lim_{t \to \infty} y(t) = 0 \).

16. Consider the equation

\[
y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0
\]

Determine the values of \( \alpha \), if any, for which all solutions tend to zero as \( t \to \infty \); also determine the values of \( \alpha \), if any, for which all (nonzero) solutions become unbounded as \( t \to \infty \).

\[\S 3.1 \ #24\]

Sol. The characteristic equation is

\[
r^2 + (3 - \alpha)r - 2(\alpha - 1) = 0
\]

Thus we get \( r = -2, \ \alpha - 1 \). Therefore the general solution is of the form

\[
y(t) = c_1 e^{-2t} + c_2 e^{(\alpha - 1)t}
\]

It is clear that if \( \alpha - 1 < 0 \Rightarrow \alpha < 1 \), then \( e^{(\alpha - 1)t} \to 0 \) as \( t \to \infty \). Thus, if \( \alpha < 1 \), then all solutions tend to zero as \( t \to \infty \). On the other hand, if \( \alpha - 1 > 0 \Rightarrow \alpha > 1 \), then \( e^{(\alpha - 1)t} \to \infty \) as \( t \to \infty \). However, for the case \( c_2 = 0 \), then \( y(t) = c_1 e^{-2t} \) for some nonzero constant \( c_1 \) and \( \lim_{t \to \infty} y(t) = 0 \). Therefore, there is no such value of \( \alpha \) for which all solutions become unbounded. \( \square \)